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Large Deviations for Expanding Transformations with Additive White Noise

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Large-deviations estimates for the autocorrelations of order k of the random process $Z_n = \phi(X_n) + \xi_n$, $n \geqslant 0$, are obtained. The processes $(X_n)_{n \geqslant 0}$ and $(\xi_n)_{n \geqslant 0}$ are independent, ξ_n, $n \geqslant 0$, are i.i.d. bounded random variables, $X_n = T^n(X_0)$, $n \in \mathbb{N}$, T: $M \to M$ is expanding leaving invariant a Gibbs measure on a compact set M, and ϕ: $M \to \mathbb{R}$ is a continuous function. A possible application of this result is the case where M is the unit circle and the Gibbs measure is the one absolutely continuous with respect to the Lebesgue measure on the circle. The case when T is a uniquely ergodic map was studied in Carmona *et al.* (1998). In the present paper T is an expanding map. However, it is possible to derive large-deviations properties for the autocorrelations samples $(1/n) \sum_{j=0}^{n-1} Z_j Z_{j+k}$. But the deviation function is quite different from the uniquely ergodic case because it is necessary to take into account the entropy of invariant measures for T as an important information. The method employed here is a combination of the variational principle of the thermodynamic formalism with Donsker and Varadhan's large-deviations approach.

KEY WORDS: Level-2 large deviations; expanding maps; Gibbs states; entropy; Markov process; additive white noise.

1. INTRODUCTION

Suppose that $(V_n)_{n \geqslant 0}$ is a random process in the probability space $(\Omega, \mathscr{F}, P)$ where each V_n takes values in a locally compact metric space S (the phase space of the process). Let $\mathscr{M}_1(S)$ be the space of probability measures on $\mathscr{B}(S)$, endowed with the weak topology; the set $\mathscr{B}(S)$ is the σ-field of the Borel subsets of S.

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The empirical means for $(V_n)_{n\geqslant 0}$ are

$$L_n(w,\cdot)=\frac{1}{n}\sum_{k=0}^{n-1}\delta_{V_k(w)}(\cdot),\qquad w\in\Omega \tag{1.1}$$

The level-2 large deviation theory for $(V_n)_{n\geqslant 0}$ deals with estimates of the type

$$\overline{\lim_{n\to+\infty}}\ \frac{1}{n}P(L_n\in F)\leqslant-\inf_{v\in F}I(v) \tag{1.2}$$

and

$$\underline{\lim_{n\to+\infty}}\ \frac{1}{n}\ln P(L_n\in G)\geqslant-\inf_{v\in G}I(v) \tag{1.3}$$

for all closed F and open G, subsets of $\mathscr{M}_1(S)$. The functional $I(v)$ is lower-semicontinuous in the weak topology; it is called "rate functional" or "level-2 entropy function." One says that the process satisfies a level-2 Large Deviation Principle (LDP) with rate functional $I(\cdot)$.

The level-1 LDP for $(V_n)_{n\geqslant 0}$ deals with means $(1/n)\sum_{k=0}^{n-1}V_k$. The analogous to (1.2) and (1.3) have F and G as subsets of S.

In a recent work, Carmona *et al.*[1] proves the existence of a level-2 LDP for a class of Markov processes $(V_n)_{n\geqslant 0}$ given by

$$V_n=(X_n,\xi_n,\xi_{n+1}),\qquad n\geqslant 0 \tag{1.4}$$

where $\xi_1,\xi_2,\ldots$ are i.i.d. random variables with common distribution η and

$$X_n=T^n(X_0) \tag{1.5}$$

T^n is the group of the iterates of a bijective uniquely ergodic transformation on the unit circle, preserving the Lebesgue measure λ, X_0 being the random variable describing the position on the circle and distributed according to λ. The processes $(X_n)_{n\geqslant 0}$ and $(\xi_n)_{n\geqslant 0}$ are independent.

In the above mentioned article, the final goal was to investigate large deviation properties of the autocorrelation of order k of the process $Z_n=\phi(X_n)+\xi_n$ where ϕ is a continuous real function on the circle. We refer the reader to A. Lopes and S. Lopes[8] for example where such kind of model appears and for general results.

By assuming $k=1$ (in order to simplify the arguments), the method employed in ref. 1 consists, as a first step (and the more difficult one), in obtaining a LDP for the empirical means of (1.4). The second step is to use the contraction principle to obtain a LDP for the autocorrelations of order 1

$$M_n = \frac{1}{n} \sum_{j=0}^{n-1} Z_j Z_{j+1}, \qquad n \geqslant 1$$

by noticing that $Z_n Z_{n+1}$ is a continuous functional of V_n. At level 1 analysis we assume that the measure η has compact support.

Under the above assumptions, the Markov process in (1.4) is ergodic in the sense of having a unique stationary distribution, which is $\lambda \times \eta \times \eta$. In this case, Carmona *et al.*[1] established a level 2 LDP for $(V_n)_{n \geqslant 0}$ in the probability space $(S^{\mathbb{N}}, \sigma(\mathscr{C}), \mathbb{P}_{xyz})$ where $S = M \times \mathbb{R} \times \mathbb{R}$ is the phase space of the process, $\sigma(\mathscr{C})$ is the σ-field of the cylinder sets of $S^{\mathbb{N}}$, and $\mathbb{P}_{xyz}(\cdot)$ is the distribution of the process when the initial distribution is $\sigma_{(x,y,z)}(\cdot)$, $(x, y, z) \in S$. The upper and lower bounds in (1.2) and (1.3), respectively, were obtained by employing Donsker and Varadhan's[3] approach.

For Markov processes having a unique stationary distribution, Donsker and Varadhan's method consists, roughly speaking, in proving that

$$I(v) = -\inf_{\phi \in \mathscr{W}} \int_S \ln \frac{\Pi\phi}{\phi}\, dv, \qquad v \in \mathscr{M}_1(S) \tag{1.6}$$

is the level 2 entropy function of the process. In (1.6)

$$\Pi\phi(v) = \int_S \phi(u)\, \Pi(v, du) \tag{1.7}$$

$\Pi(v, du)$ is the transition function given by

$$\Pi((x, y, z), d(x_1, y_1, z_1)) = \delta_{T(x)}(dx_1)\, \delta_z(dy_1)\, \eta(dz_1) \tag{1.8}$$

and

$$\begin{aligned} \mathscr{W} = \{\phi\colon S \to \mathbb{R} : \phi \text{ is continuous, } \exists a, b \text{ such that} \\ 0 < a \leqslant \phi(v) \leqslant b < +\infty,\ \forall v \in S\} \end{aligned} \tag{1.9}$$

In the present paper we use the same approach as in ref. 1 but now assuming that T is not uniquely ergodic anymore. Consequently, $(V_n)_{n \geqslant 0}$ is a Markov process with more than one stationary measure and the natural question to ask for is in what extent is still possible to have large deviations results. Since the lack of ergodicity of V_n comes from X_n, we must specify the class of transformations $T\colon M \to M$ we are going to deal with as well as the class of initial distributions of $(V_n)_{n \geqslant 0}$ for which we are able to derive large deviations estimates.

The concept of "equilibrium state" or "Gibbs state" is closely related with the question raised above. One says that a T-invariant probability measure μ_ψ is an equilibrium state for the continuous function $\psi: M \to \mathbb{R}$ if it attains the supremum

$$\mathscr{P}(\psi) \equiv \sup_{\mu \in \mathscr{M}^T(M)} \left\{ \int \psi \, d\mu + h_\mu(T) \right\} \tag{1.10}$$

where $\mathscr{M}^T(M)$ is the class of all T-invariant probability measures with support M, $h_\mu(T)$ is the Kolmogorov-Sinai entropy (see ref. 12) of the map T with respect to the T-invariant probability measure μ. The functional $\mathscr{P}(\cdot)$ is called the "topological pressure" of ψ (see refs. 12 and 9). It is worth to point out that, in the case of large deviations of a bijective uniquely ergodic transformation T and an absolutely continuous invariant measure of the circle, Kolmogorov-Sinai entropy does not play any role. Note that there exists uniquely ergodic transformations (not bijective) with positive entropy.

The existence of equilibrium states for any $\psi \in C(M)$, $C(M)$ being the space of real continuous functions in M, is a consequence of the upper-semicontinuity of $h_\mu(T)$ as a function of μ. However, an equilibrium state μ_ψ is not unique for many continuous functions (see ref. 12). For expanding maps (therefore, not bijective anymore), the uniqueness of μ_ψ holds for Holder continuous functions. A continuous transformation T in a compact metric space M is an expanding map, if there exists $\varepsilon > 0$ and $1 < \lambda < +\infty$ such that for all x, y with $d(x, y) < \varepsilon$, $d(T(x), T(y)) \geqslant \lambda d(x, y)$. Relevant examples of expanding maps are the shift in the Bernoulli space $\{1, 2,..., d\}^{\mathbb{N}}$ and one-dimensional maps obtained from the foliations of certain hyperbolic dynamical systems (see refs. 7, 9, and 11).

By using the variational principle of Thermodynamic formalism, Kifer[5] and Lopes[6] proved relations (1.2) and (1.3) for the empirical means

$$\zeta_x^n = \frac{1}{n} \sum_{k=0}^{n-1} \delta_{T^k(x)}, \qquad x \in M \tag{1.11}$$

with $P \equiv m \in \mathscr{M}_1(M)$ satisfying some conditions to be specified later. Notice that m is the distribution of X_0.

In Kifer's[5] approach, roughly speaking, this method consists in proving that

$$\mathscr{Q}_m(\psi) \equiv \lim_{n \to +\infty} \frac{1}{n} \ln \int_M \exp\left[n \int \psi \, d\zeta_x^n \right] dm(x) \tag{1.12}$$

exists for all continuous function ψ; $\mathcal{Q}_m(\psi)$ is called the "m-generalized pressure" of ψ. The next step is to show that the level-2 entropy function is the Legendre transform of the generalized pressure in the space $\mathcal{M}_1(M)$. It is possible to show that the topological pressure $\mathcal{P}(\psi)$ is equal to $\mathcal{Q}_m(\psi)$ when m is the "maximal entropy measure" (the one that maximizes the entropy $h_\mu(T)$); this measure may also be seen as the equilibrium state of $\psi \equiv 0$.

Kifer assumed that m satisfies condition

(K1) $\exists \varphi \in C(M), \forall \delta > 0, \forall n > 0, \forall x \in M,$

$$(A_\delta(n))^{-1} \leqslant m(\mathcal{U}_\delta(x, n, M)) \exp\left\{-n \int_M \varphi \, d\zeta_x^n\right\} \leqslant A_\delta(n)$$

where $A_\delta(n) > 0$, $\lim_{n \to +\infty} (1/n) \ln A_\delta(n) = 0$ and

$$\mathcal{U}_\delta(x, n, M) = \{ y \in M : d(T^j(x), T^j(y)) < \delta, \forall 0 \leqslant j \leqslant n-1 \}.$$

Note that m is coupled to the chosen φ. If m satisfies condition (K_1) and it is a T-invariant probability measure, then $\int \varphi \, dm + h_m(T) = 0$ whence m is an equilibrium state for φ and $\mathcal{P}(\varphi) = 0$ (see basic properties of equilibrium states in ref. 9, Chapter 3).

In Proposition 3.2,[5] it was proved that, for m satisfying (K1) and for any $\psi \in C(M)$,

$$\mathcal{Q}_m(\psi) \equiv \lim_{n \to +\infty} \frac{1}{n} \ln \int_M \exp\left\{n \int_M \psi \, d\zeta_x^n\right\} dm(x) = \mathcal{P}(\varphi + \psi) \tag{1.13}$$

where $\mathcal{P}(\cdot)$ was introduced in (1.10).

The operator $\mathcal{Q}_m(\cdot)$ is convex and weakly continuous. Its Legendre transform $I_m(\mu)$, $\mu \in \mathcal{M}_1(M)$, is convex and weakly lower-semicontinuous. From the duality property,

$$\mathcal{Q}_m(\psi) = \sup_{\mu \in \mathcal{M}_1(M)} \left(\int \psi \, d\mu - I_m(\mu) \right), \qquad \psi \in C(M) \tag{1.14}$$

Hence, the definitions of $\mathcal{P}(\cdot)$ in (1.10) implies that

$$I_m(\mu) = \begin{cases} -\int \varphi \, d\mu - h_\mu(T), & \text{if} \quad \mu \in \mathcal{M}^T(M) \\ +\infty, & \text{otherwise} \end{cases} \tag{1.15}$$

where φ is the one associated with m in (K1). Notice that, if $m \in \mathcal{M}^T(M)$ then $I_m(m) = 0$.

Since $\mathcal{M}_1(M)$ is compact and $I_m(\cdot)$ is weakly lower-semicontinuous (T is such that $h_\mu(T)$ is upper-semicontinuous) then, for each $\psi \in C(M)$, there exists $\mu_{\psi,m} \in \mathcal{M}^T(M)$ such that $\mathcal{Q}_m(\psi) = \int \psi \, d\mu_{\psi,m} - I_m(\mu_{\psi,m})$ or, equivalently,

$$\mathcal{P}(\varphi + \psi) = \int (\varphi + \psi) \, d\mu_{\psi,m} + h_{\mu_{\psi,m}}(T) \tag{1.16}$$

Such measure $\mu_{\psi,m}$ is called an equilibrium state for ψ with respect to the generalized pressure $\mathcal{Q}_m(\cdot)$. Note that such measure depends on ψ and m. When m is the maximal entropy measure, then $\mu_{\psi,m} = \mu_\psi$ according to the previous notation of equilibrium state associated to ψ.

In the variational method, the existence of the limit in (1.12) is enough for obtaining the upper bound (1.2) with rate functional $I_m(\cdot)$. The lower bound is more delicate: assumptions on the uniqueness of equilibrium states are necessary.

Lopes[6, 7] studied, specifically, level 1 and level 2 large deviations for $(X_n)_{n\geqslant 0}$ when M is compact, T is an expanding map of degree d and X_0 is distributed according to the "maximal entropy measure;" it is a special case of Gibbs measure (ψ is constant). Among the possible invariant measures for an expanding map T of the circle there is one that is absolutely continuous with respect to the Lebesgue measure (see ref. 9 for reference). This measure is sometimes called SBR measure and it is mixing. In most of the cases of expanding maps the maximal entropy measure is not the SBR measure.

In this paper, in order to study level-2 large deviations for $(V_n)_{n\geqslant 0}$ in (1.4), the main assumptions are that T has upper-semicontinuous entropy and satisfies condition

(K2) $T: M \to M$ is such that for each $\mu \in \mathcal{M}^T(M)$, for all, $\delta > 0$, there exists a T-invariant probability measure μ_1, weakly close to μ, and a Holder continuous function $\tilde{\psi}$ such that

$$h_{\mu_1}(T) > h_\mu(T) - \delta$$

and μ_1 is the unique solution of

$$h_\nu(T) + \int_M \tilde{\psi} \, d\nu = \mathcal{P}(\tilde{\psi}).$$

Moreover, $m \in \mathcal{M}_1(M)$ satisfies condition (K1), and the equilibrium states with respect to the generalized pressure $\mathcal{Q}_m(\cdot)$ are unique for Holder continuous functions. The initial distributions for $(V_n)_{n\geqslant 0}$ are assumed to be of the type $m \times \delta_{(y,z)}$ with $y, z \in \mathbb{R}$.

It is worth to remark that expanding maps satisfy the above assumptions as well as condition (K_2) (see Theorem 8 in ref. 7). So, in this paper, one can regard $T: M \to M$ as expanding. The method employed here is a mixture of the variational principle and Donsker and Varadhan's approach.

Let $\mathbb{P}_\mu(\cdot)$, $\mu \in \mathscr{M}_1(S)$, be the probability measure on the measurable space $(S^{\mathbb{N}}, \sigma(\mathscr{C}))$ induced by the transition function Π in (1.8). The measure μ is the initial distribution of the Markov process $(V_n)_{n \geqslant 0}$. In particular, if $\mu = m \times \delta_{(y, z)}$ we have $\mathbb{P}_\mu = \mathbb{P}_{m, y, z}$. The corresponding expectation is denoted by $\mathbb{E}_{m, y, z}$.

The main result in this paper is the level-2 Large Deviation Principle for $(V_n)_{n \geqslant 0}$ in the probability space $(S^{\mathbb{N}}, \sigma(\mathscr{C}), \mathbb{P}_{m, y, z}(\cdot))$. The explicit form for the rate functional is

$$I^m(v) = I_m(\pi_1 v) + I(v) \tag{1.17}$$

$I_m(\cdot)$ given in (1.15) and $I(\cdot)$ is the functional in (1.6) which, in Section 3, is proved to be equal to

$$I(v) = \begin{cases} \int_S \ln \dfrac{a}{a_{12}}\, dv, & v \in \mathscr{M}_0 \\ +\infty, & \text{otherwise} \end{cases} \tag{1.18}$$

where

$$\mathscr{M}_0 = \Big\{ v \in \mathscr{M}_1(S) : \pi_1 v \in \mathscr{M}^T(M),\ v \ll \pi_1 v \times \eta \times \eta, \tag{1.19}$$

$$\pi_{12} v = \pi_{13} v T^{-1},\ \int_S |\ln a/a_{12}|\, dv < +\infty \Big\}$$

$$a(x, y, z) = \frac{dv}{d(\pi_1 v \times \eta \times \eta)}(x, y, z) \qquad a_{12}(x, y) = \int_{\mathbb{R}} a(x, y, z)\, d\eta(z)$$

$\pi_i v$, $\pi_{ij} v$ are the i- and ij-marginals of v, and

$$v T^{-1}(A \times B \times C) = v(T^{-1}(A) \times B \times C),$$

for any measurable rectangle $A \times B \times C$.

It follows from uniqueness of equilibrium states that when $m \in \mathscr{M}_0^T(M)$ and φ is Holder continuous then the infimum of $I_m(\cdot)$ in a compact set not containing $\mu \times \eta \times \eta$ is positive (see Lemma 3.3(b)). Therefore, the probability of such compact sets decays exponentially fast.

In Section 2 we prove the upper bound by taking into account the limit in (1.13) and the fact that $(V_n)_{n \geqslant 0}$ is a Markov process. We use Donsker and Varadhan's[3] approach but keeping in mind that the process is not uniquely ergodic (has more than one stationary distribution).

In Section 3 we prove the lower bound. The main property we shall use there is condition (K2). We also use results contained in ref. 1; again Donsker and Varadhan's method is used.

2. UPPER BOUND

In this paragraph we shall prove the upper bound in (1.2) for the process $(V_n)_{n \geqslant 0}$ in (1.4). The map $T: M \to M$ is assumed to be expansive and M a compact metric space.

Clearly, for each n,

$$L_n(w, \cdot) = \frac{1}{n} \sum_{k=0}^{n-1} \delta_{(X_k, \xi_k, \xi_{k+1})(w)}(\cdot), \qquad w \in S^{\mathbb{N}} \tag{2.1}$$

is a random probability measure in the space $(S^{\mathbb{N}}, \sigma(\mathscr{C}))$. In this space we shall consider the family of probability measures $\mathbb{P}_{m, y, z}(\cdot)$, with y, $z \in \mathbb{R}$, and $m \in \mathscr{M}_1(M)$ satisfying condition (K1).

For each $w \in S^{\mathbb{N}}$, $L_n(w, \cdot) \in \mathscr{M}_1(S)$. Let $Q^n_{m, y, z}(\cdot)$ be the distribution of L_n on $\mathscr{B}(\mathscr{M}_1(S))$, i.e.,

$$Q^n_{m, y, z}(A) = \mathbb{P}_{m, y, z}[L_n \in A], \qquad A \in \mathscr{B}(\mathscr{M}_1(S)). \tag{2.2}$$

Theorem 2.1. Let $m \in \mathscr{M}_1(M)$ satisfy condition (K1). Then, for any closed $F \subset \mathscr{M}_1(S)$,

$$\overline{\lim_{n \to +\infty}} \frac{1}{n} \ln Q^n_{m, y, z}(F) \leqslant - \inf_{v \in F} I^m(v)$$

where

$$I^m(v) = \begin{cases} -\int \varphi \, d\pi_1 v - h_{\pi_1 v}(T) + \int_S \ln \dfrac{a}{a_{12}} \, dv, & v \in \mathscr{M}_0 \\ + \infty, & \text{otherwise} \end{cases} \tag{2.3}$$

$\mathscr{M}_0$ being the set in (1.19). The function $h_{\pi_1 v}(T)$ is the Kolmogorov-Sinai entropy of the T-invariant probability measure $\pi_1 v$.

Proof. Let $\phi \in \mathscr{W}$ (this set is defined in (1.9)) and $\psi \in C(M)$. Let $e^{-W} = \phi/u$, $u = \Pi\phi$. Let $\mathbb{E}^{Q^n_{m,\,y,\,z}}$ be the expectation associated to the distribution introduced in (2.2). Then

$$\begin{aligned} &\mathbb{E}^{Q^n_{m,\,y,\,z}}\left[\exp\left\{-n\int_S (W-\psi)\,d\mu\right\}\right] \\ &\quad = \mathbb{E}_{m,\,y,\,z}\left[\exp\left\{-n\int_S (W-\psi)\,dL_n(w, d(x_1, y_1, z_1))\right\}\right] \\ &\quad = \int_M \exp\left\{\sum_{k=0}^{n-1}\psi(T^k(x))\right\}\mathbb{E}_{xyz}\exp\left(-\sum_{k=0}^{n-1} W(V_k)\right)dm(x) \end{aligned}$$

The Markov property implies that

$$\mathbb{E}_v\{\exp(-[W(V_0)+\cdots+W(V_{n-1})])\,\Pi V_{n-1}\} = \phi(v), \qquad \forall v \in S, \quad n \geqslant 1$$

Since $\phi \in \mathscr{W}$, $\exists K > 0$ such that

$$\mathbb{E}_v\left[\exp\left\{-\sum_{k=0}^{n-1} W(V_k)\right\}\right] \leqslant K$$

and we get

$$\begin{aligned} &\mathbb{E}_{m,\,y,\,z}\left[\exp\left\{\sum_{k=0}^{n-1}\psi(X_k)\right\}\exp\left\{-\sum_{k=0}^{n-1}W(V_k)\right\}\right] \\ &\quad \leqslant K\int_M \exp\left\{\sum_{k=0}^{n-1}\psi(T^k(x))\right\}dm(x) \end{aligned}$$

Then, for $F \subset \mathscr{M}_1(S)$,

$$\begin{aligned} &\exp\left\{-n\sup_{\mu \in F}\left(\int_S (W-\psi)\,d\mu\right)\right\} Q^n_{m,\,y,\,z}(F) \\ &\quad \leqslant K\int_M \exp\left\{\sum_{k=0}^{n-1}\psi(T^k(x))\right\}dm(x) \end{aligned}$$

and, by taking into account (1.13), we may write

$$\overline{\lim_{n \to +\infty}}\ \frac{1}{n}\ln Q^n_{m,\,y,\,s}(F) \leqslant \sup_{\mu \in F}\left(\int_S (W-\psi)\,d\mu\right) + \mathscr{Q}_m(\psi) \tag{2.4}$$

The above inequality holds for all $\phi \in \mathscr{W}$ and $\psi \in C(M)$. Then

$$\overline{\lim_{n\to+\infty}} \frac{1}{n} \ln Q^n_{m,y,z}(F) \leqslant \inf_{\phi,\psi} \left[\sup_{\mu\in F} \int_S (W-\psi)\, d\mu + \mathscr{Q}_m(\psi) \right]$$

Up to now, F is any measurable subset of $\mathscr{M}_1(S)$. Let $F \subset \bigcup_{i=1}^k F_i$, F_i measurable. In this case

$$\overline{\lim_{n\to+\infty}} \frac{1}{n} \ln Q^n_{m,y,z}(F)$$

$$\leqslant \inf_{\substack{F_1,\dots,F_k \\ F\subset \bigcup_{i=1}^k F_i}} \sup_{1\leqslant i\leqslant k} \inf_{\phi,\psi} \left[\sup_{\mu\in F_i} \int_S (W-\psi)\, d\mu + \mathscr{Q}_m(\psi) \right]$$

If F is compact, one can prove as in ref. 2, where a closely related situation is considered, that the right hand side of the above inequality is less or equal to

$$\sup_{\mu\in F} \inf_{\phi,\psi} \left[\int_S (W-\psi)\, d\mu + \mathscr{Q}_m(\psi) \right]$$

On the other hand,

$$\sup_{\mu\in F} \inf_{\phi,\psi} \left[\int_S (W-\psi)\, d\mu + \mathscr{Q}_m(\psi) \right]$$

$$= -\inf_{\mu\in F} \left\{ -\inf_{\phi} \int_S W\, d\mu + \sup_{\psi} \left[\int \psi\, d\pi_1 \mu - \mathscr{Q}_m(\psi) \right] \right\}$$

$$= -\inf_{\mu\in f} \left[I(\mu) + I_m(\pi_1 \mu) \right]$$

where $I_m(\cdot)$ and $I(\cdot)$ are defined in (1.15) and (1.6) (or (1.18)), respectively. Therefore, we have proved the upper bound for compact sets.

For going from compact to closed sets, we observe that the family of distributions in (2.2) is exponentially tight because, for all $x \in M$, $(Q^n_{xyz}(\cdot))_{n\geqslant 1}$ is exponentially tight (see Lemma 4.1 in ref. 1) and

$$Q^n_{m,y,z}(A) = \mathbb{E}_m(Q_{xyz}(A)), \qquad A \in \mathscr{B}(\mathscr{M}_1(S)) \quad \blacksquare$$

3. LOWER BOUND

In this part we shall prove the inequality in (1.3) for the process $(V_n)_{n\geqslant 0}$ in (1.4) satisfying the conditions specified in Section 1. The proof of the lower bound is much more delicate. We shall use the uniqueness condition for equilibrium states given by (K2) as well as Donsker and Varadhan's approach[3] as was employed by Carmona *et al.*[1]

To obtain the estimate from below for $\mathbb{P}_{m,y,z}[L_n \in S(\mu, \varepsilon)]$, where $S(\mu, \varepsilon)$ is an open sphere contained in $\mathcal{M}_1(S)$, we shall use a "generalized Cramér transformation": we construct a measure μ^n absolutely continuous with respect to m, a functional $\Phi_\mu(V_n: n \geqslant 0)$, and a new probability $\mathbb{P}'_{\mu^n,y,z}$ given by

$$\mathbb{P}'_{\mu^n,y,z}(A) = \int_A \Phi_\mu(w)\, d\mathbb{P}_{m,y,z}(w), \qquad A \in \sigma(\mathscr{C})$$

such that, with large $\mathbb{P}'_{\mu^n,y,z}$-probability, the process L_n, $n \geqslant 0$ falls in a neighborhood of μ.

Let us define

$$\tilde{I}(\nu) = -\inf_{\phi \in \mathscr{W}} \int_S \ln \frac{\Pi\phi}{\phi}\, d\nu, \qquad \nu \in \mathcal{M}_1(S) \tag{3.1}$$

where $\mathscr{W}$ is the set in (1.9) and $\Pi\phi$ is defined in (1.7). For convenience, we have changed the notation of the functional in (1.6).

The functionals $I_m(\cdot)$ and $I(\cdot)$ are defined in (1.15) and (1.18), respectively. The set $\mathcal{M}_0$ is given in (1.19).

The proof of the following lemma is similar to the proof of Lemma 3.2 in ref. 1 and it is omitted.

Lemma 3.1. (a) If $\nu \in \mathcal{M}_0$ then $\tilde{I}(\nu) < +\infty$.

(b) If $\tilde{I}(\nu) < +\infty$ and $\pi_1 \nu \in \mathcal{M}^T(M)$ then $\nu \in \mathcal{M}_0$.

(c) $\tilde{I}(\nu) = I(\nu)$.

Lemma 3.2. The stationary distributions for the Markov process $(V_n)_{n \geqslant 0}$ in (1.4) are of the type $\nu = \mu \times \eta \times \eta$ with $\mu \in \mathcal{M}^T(M)$.

Proof. Similarly to the proof of Lemma 2.5 in ref. 3, one can show that $I(\nu) = 0$ if and only if ν is a stationary distribution for $(V_n)_{n \geqslant 0}$.

Let $\nu = \mu \times \eta \times \eta$, $\mu \in \mathcal{M}^T(M)$. Clearly $I(\nu) = 0$. Then ν is stationary for $(V_n)_{n \geqslant 0}$.

Let $\nu \in \mathcal{M}_1(S)$ be stationary for $(V_n)_{n \geqslant 0}$. Then $I(\nu) = 0 < +\infty$ and $\pi_1 \nu \in \mathcal{M}^T(M)$. Hence, Lemma 3.1 implies that $\nu \in \mathcal{M}_0$. Also, for all measurable rectangle $A \times B \times C$,

$$\int_S \Pi((x, y, z), A \times B \times C)\, d\nu(x, y, z) = \nu(A \times B \times C)$$

However, for any ν,

$$\int_S \Pi((x, y, z), A \times B \times C)\, d\nu(s, y, z) = \pi_{13}\nu(T^{-1}(A) \times B)\, \eta(C)$$

Since $\nu \in \mathcal{M}_0$ then $\pi_{12}\nu = \pi_{13}\nu T^{-1}$ so that the right hand side of the above equality is equal to $\pi_{12}\nu(A \times B)\, \eta(C)$. Hence, the stationarity of ν implies that

$$\nu(A \times B \times C) = \pi_{12}\nu(A \times B)\, \eta(C)$$

Therefore, $\nu = \mu \times \eta$ with $\mu \in \mathcal{M}_1(M \times \mathbb{R})$ and $\pi_1\nu = \pi_1\mu \in \mathcal{M}^T(M)$. We shall prove that $\mu = \pi_1\nu \times \eta$.

Since $\nu = \mu \times \eta$ we have

$$\pi_{13}\nu(A \times B) = \nu(A \times \mathbb{R} \times B) = \mu(A \times \mathbb{R})\, \eta(B) = \pi_1\mu(A)\, \eta(B)$$

that is, $\pi_{13}\nu = \pi_1\mu \times \eta$. Similarly, one can show that $\pi_{13}\nu T^{-1} = \pi_1\mu \times \eta$. Since $\pi_{12}\nu = \pi_{13}\nu T^{-1}$ we have $\pi_1 2\nu = \pi_{13}\nu = \pi_1\nu \times \eta$. Therefore, $\nu = \mu \times \eta$ is of the type $\pi_1\nu \times \eta \times \eta$ with $\pi_1\nu \in \mathcal{M}^T(M)$. ▮

The following lemma is a consequence of Lemma 3.1 and Lemma 3.2.

Lemma 3.3. Let $m \in \mathcal{M}_1(M)$ satisfy condition (K1) and φ the continuous function associated to m. Then

(a) $I^m(\nu) < +\infty$ if and only if $\nu \in \mathcal{M}_0$.

(b) $I(\nu) = 0$ if and only if $\nu = \mu \times \eta \times \eta$ with $\mu \in \mathcal{M}^T(M)$. Moreover, if $m \in \mathcal{M}^T(M)$ and φ is Holder continuous then $I^m(\nu) = 0$ if and only if $\nu = m \times \eta \times \eta$.

Let $\mu \in \mathcal{M}^T(M)$ and $\tilde{\psi}$ and μ_1 as in condition (K2). Take $\psi = \tilde{\psi} - \varphi$ where φ is the continuous function in condition (K1) associated to m. We have

$$\mathcal{P}(\tilde{\psi}) = \mathcal{P}(\tilde{\phi} - \varphi - \varphi) = \mathcal{P}(\varphi + \psi)$$

It follows from condition (K2) that

$$\mathcal{P}(\varphi + \psi) = \int_M (\varphi + \psi)\, d\mu_1 + h_{\mu_1}(T)$$

Proposition 3.1. For any $\nu \in \mathcal{M}_0$ there exists $\mu \in \mathcal{M}_0$, weakly close to ν, and

$$I^m(\nu) > I^m(\mu) - \varepsilon, \qquad \forall \varepsilon > 0 \tag{3.2}$$

Proof. Let $\nu \in \mathcal{M}_0$ and $a(x, y, z) = (d\nu)/(d\pi_1 \nu \times \eta \times \eta)(x, y, z)$. Let $\varepsilon > 0$. Since $\pi_1 \nu \in \mathcal{M}^T(M)$, condition (K2) says that there exists a T-invariant probability measure μ_1, weakly close to $\pi_1 \nu$, and a Holder continuous function $\tilde{\psi}$ in M such that

$$h_{\mu_1}(T) > h_{\pi_1 \nu}(T) - \frac{\varepsilon}{3} \tag{3.3}$$

and μ_1 is the unique solution of

$$h_\nu(T) + \int_M \tilde{\psi} \, d\nu = \mathscr{P}(\tilde{\psi})$$

Take $\mu \in \mathcal{M}_1(S)$ such that $\pi_1 \mu = \mu_1$ and $\mu \ll \pi_1 \mu \times \eta \times \eta$. Define $\tilde{\psi}(x, y, z) \equiv \tilde{\psi}(x)$. Clearly $\tilde{\psi}$ is continuous and bounded in S. Consequently, by writing $\tilde{\psi}(x) = (\varphi + \psi)(x)$, the operator

$$\Psi(\nu) = \int_S \psi(x, y, z) \, d\nu = \int_M [\tilde{\psi}(x) - \varphi(x)] \, d\pi_1 \nu(x)$$

is weakly continuous. Then there exists a Lévy neighborhood N^ε_μ such that, for any $\tilde{\mu} \in N^\varepsilon_\mu$,

$$\left| \int_S \psi(x, y, z) \, d\mu(x, y, z) - \int_S \psi(x, y, z) \, d\tilde{\mu}(x, y; z) \right| < \frac{\varepsilon}{3}$$

Since $\mu \ll \pi_1 \mu \times \eta \times \eta$ then

$$\int_S \psi(x, y, z) \, d\mu(x, y, z) = \int_M \psi(x) \, d\pi_1 \mu(x) = \int_M \psi(x) \, d\mu_1(x)$$

Similarly,

$$\int_{S'} \psi(x, y, z) \, d\nu(x, y, z) = \int_M \psi(x) \, d\pi_1 \nu(x).$$

Hence $\nu \in N^\varepsilon_\mu$ because μ_1 is weakly close to $\pi_1 \nu$.

The measure μ can be chosen in $\mathscr{M}_0$ by assuming that $\pi_{12}\mu = \pi_{13}\mu T^{-1}$. The lower-semicontinuity of $I(\cdot)$ and the fact that $\nu \in N_\mu^\varepsilon$ implies that

$$I(\nu) > I(\mu) - \frac{\varepsilon}{3}$$

Moreover,

$$\int_M \varphi \, d\pi_1 \nu < \int \varphi \, d\mu_1 + \frac{\varepsilon}{3}$$

Recalling the definition of $I^m(\cdot)$ in (1.17) and the relation in (3.3) we get (3.2). ∎

Let us define

$$\mathscr{M}_2 = \Big\{ \nu \in \mathscr{M}_0 : \frac{d\nu}{d\pi_1 \nu \times \eta \times \eta}(x, y, z) = b(x, y, z) \text{ such that}$$

$$\exists c, d \text{ with } 0 < c \leqslant b(x, y, z) \leqslant d < +\infty, \forall (x, y, z) \in S \Big\} \tag{3.4}$$

As in Lemma 2.9 in ref. 3 one can prove that for G an open subset of $\mathscr{M}_1(S)$,

$$\inf_{\nu \in G} I^m(\nu) = \inf_{\nu \in G \cup \mathscr{M}_2} I^m(\nu) \tag{3.5}$$

Fix $\mu \in \mathscr{M}_2$ with $(d\mu)/(d\pi_1 \mu \times \eta \times \eta)(x, y, z) = a(x, y, z)$ and let

$$a_{12}(x, y) = \int_{\mathbb{R}} a(x, y, z) \, d\eta(z)$$

Define

$$\Pi'((x, y, z), d(x_1, y_1, z_1)) = \frac{a(x_1, y_1, z_1)}{a_{12}(x_1, y_1)} \Pi((x, y, z), d(x_1, y_1, z_1)) \tag{3.6}$$

where Π is the transition function of $(V_n)_{n \geqslant 0}$ introduced in (1.8). Associated to Π' we define the functional

$$I'(\nu) = -\inf_{\phi \in \mathscr{W}} \int_S \ln \frac{\Pi' \phi}{\phi} \, d\nu, \qquad \nu \in \mathscr{M}_1(S) \tag{3.7}$$

Proposition 3.2.

$$I'(v)=\begin{cases}\int_S \ln\frac{b}{b_{12}}\,dv-\int_S \ln\frac{a}{a_{12}}\,dv, & \text{if}\quad v\in\mathscr{M}_0\\ +\infty, & \text{otherwise}\end{cases}\tag{3.8}$$

where

$$b(x,y,z)=\frac{dv}{d\pi_1\, v\eta\times\eta}(x,y,z),\qquad b_{12}(x,y)=\int_{\mathbb{R}} b(x,y,z)\,d\eta(z).$$

Proof. For $\phi\in\mathscr{W}$

$$\begin{aligned}\Pi'\phi(x,y,z)&\equiv\int_S \phi(x_1,y_1,z_1)\,\Pi'((x,y,z),d(x_1,y_1,z_1))\\ &=\int_S \phi(x_1,y_1,z_1)\frac{a(x_1,y_1,z_1)}{a_{12}(x_1,y_1)}\,\Pi((x,y,z),d(x_1,y_1,z_1))\\ &=\Pi(\phi\tilde{a})(x,y,z)\end{aligned}$$

with $\tilde{a}(x,y,z)=a(x,y,z)/a_{12}(x,y)$.

Let $\mathscr{W}^*$ be the set of all nonnegative measurable functions in S, bounded away from zero and infinity. Then

$$\begin{aligned}I'(v)&=-\inf_{\phi\in\mathscr{W}}\int_S \ln\frac{\Pi(\phi\tilde{a})}{\phi}\,dv\\ &=-\inf_{\phi\in\mathscr{W}}\int_S \ln\frac{\Pi(\phi\tilde{a})}{\phi\tilde{a}}\,dv-\int_S \ln\tilde{a}\,dv\end{aligned}\tag{3.9}$$

Since $\phi\tilde{a}\in\mathscr{W}^*$,

$$\inf_{\phi\in\mathscr{W}}\int_S \ln\frac{\Pi(\phi\tilde{a})}{\phi\tilde{a}}\,dv\leqslant\inf_{\phi\in\mathscr{W}^*}\int_S \ln\frac{\Pi(\phi)}{\phi}\,dv$$

On the other hand, given $\delta>0$, there exists $\tilde{\phi}\mathscr{W}^*$ such that

$$\int_S \ln\frac{\Pi\tilde{\phi}}{\tilde{\phi}}\,dv<\inf_{\phi\in\mathscr{W}^*}\int_S \ln\frac{\Pi\phi}{\phi}\,dv+\delta$$

Let $\tilde{\tilde{\phi}} = \tilde{\phi}/\tilde{a}$. Then $\tilde{\tilde{\phi}} \in \mathscr{W}^*$ and $\Pi\tilde{\phi}/\tilde{\phi} = \Pi(\tilde{\tilde{\phi}}\tilde{a})/\tilde{\tilde{\phi}}\tilde{a}$. Hence,

$$\inf_{\phi \in \mathscr{W}^*} \int_S \ln \frac{\Pi\phi\tilde{a}}{\phi\tilde{a}}\, dv \leqslant \inf_{\phi \in \mathscr{W}^*} \int_S \ln \frac{\Phi\phi}{\phi}\, dv$$

But, from Lusin's theorem (see ref. 10), the above infimum in $\mathscr{W}^*$ is the same as in $\mathscr{W}$. Then,

$$\inf_{\phi \in \mathscr{W}} \int_S \ln \frac{\Pi\phi\tilde{a}}{\phi\tilde{a}}\, dv = \inf_{\phi \in \mathscr{W}} \int_S \ln \frac{\Pi\phi}{\phi}\, dv$$

Returning to (3.9), we get

$$I'(v) = I(v) - \int_S \ln \tilde{a}\, dv$$

Moreover, according to Lemma 1, $I(v) < +\infty$ if and only if $v \in \mathscr{M}_0$ and the result follows. ∎

Lemma 3.4. Let $v \in \mathscr{M}_1(S)$. Then v is invariant for Π' if and only if $v \in \mathscr{M}_0$ and $(dv)/(d\pi_1\, v \times \eta \times \eta)(x, y, z) = a(x, y, z)$.

Proof. As in Lemma 2.5 in ref. 3, one can prove that $I'(v) = 0$ if and only if v is invariant for Π'.

Let v be invariant for Π'. Then $I'(v) = 0$ and $v \in \mathscr{M}_0$. Let $A \times B \times C$ be a measurable rectangle. Then

$$\int_S \Pi'((x, y, z), A \times B \times C)\, dv(x, y; z)$$

$$= \int_{A \times B \times C} \frac{a(x, z, z_1)}{a_{12}(x, y)}\, d\eta(z_1)\, d\pi_{13}\, v(T^{-1}(x), z)$$

$$= \int_{A \times B \times C} \frac{a(x, z, z_1)}{a_{12}(x, y)}\, d\eta(z_1)\, d\pi_{12}\, v(x, z)\, d\eta(z_1)$$

$$= \int_{A \times B \times C} a(x, y, z)\, d\pi_1\, v(x)\, d\eta(y)\, d\eta(z)$$

and the result follows. ∎

Theorem 3.1 (Lower Bound). Let G be open in $\mathscr{M}_1(S)$. Then

$$\underline{\lim}_{n \to +\infty} \frac{1}{n} \ln Q^n_{m, y, z}(G) \geqslant - \inf_{v \in G} I^m(v)$$

Proof. Let $G \subset \mathscr{M}_1(S)$ be open and $v \in G \cap \mathscr{M}_2$. From Proposition 3.1, $\forall \varepsilon > 0$, $\exists \mu \in \mathscr{M}_2$, $\exists \psi \in C(M)$, and a Lévy's neighborhood $N^\varepsilon_\mu \equiv S(\mu, \varepsilon)$ such that for any $\tilde{\mu} \in S(\mu, \varepsilon)$,

$$\left| \int_X \psi(x, y, z)\, d\mu(x, y, z) - \int_S \psi(x, y, z)\, d\tilde{\mu}(x, y, z) \right| < \varepsilon$$

where $\psi(x, y, z) \equiv \psi(x)$. Moreover, $v \in S(\mu, \varepsilon)$. Since G is open and $v \in G$, one can choose $\varepsilon > 0$ sufficiently small such that $S(\mu, \varepsilon) \subset G$.

Define $\mu^n \in \mathscr{M}_1(M)$ by

$$\frac{d\mu^n}{dm}(x) = \frac{\exp\{\sum_{k=0}^{n-1} \psi(T^k(x))\}}{\exp\{n\mathscr{Q}^n(\varphi + \psi)\}}, \qquad x \in M \tag{3.10}$$

where $\psi \in C(M)$ can be taken as $\psi = \tilde{\psi} - \varphi$, $\tilde{\psi}$ the function from condition (K2) (see proof of Proposition 3.1) and

$$\mathscr{Q}^n(\varphi + \psi) = \frac{1}{n} \ln \int_M \exp\left\{\sum_{k=0}^{n-1} \psi(T^k(x))\right\} dm(x) \tag{3.11}$$

Notice that

$$\frac{dm}{d\mu^n}(x) = \exp\{n\mathscr{Q}^n(\varphi + \psi)\} \cdot \exp\left\{-\sum_{k=0}^{n-1} \psi(T^k(x))\right\} \tag{3.12}$$

Let us introduce the sets

$$E^\varepsilon_{n, \mu} = [L_n \in S(\mu, \varepsilon)] \tag{3.13}$$

where L_n is defined in (2.1). We have

$$\begin{aligned} Q^n_{m, y, z}(S(\mu, \varepsilon)) &= \mathbb{P}_{m, y, z}(E^\varepsilon_{n, \mu}) = \int_M \mathbb{P}_{xyz}(E^\varepsilon_{n, \mu})\, dm(x) \\ &= \int_M \exp\{n\mathscr{Q}^n(\varphi + \psi)\} \\ &\qquad \times \exp\left\{-\sum_{k=0}^{n-1} \psi(T^k(x))\right\} \mathbb{P}_{xyz}(E^\varepsilon_{n, \mu})\, d\mu^n(x) \end{aligned} \tag{3.14}$$

Define $\mathbb{P}'_{xyz}(\cdot)$ as the probability measure on $(S^{\mathbb{N}}, \sigma(\mathscr{C}))$ induced by the transition function Π'. We have

$$\mathbb{P}_{xyz}(E^{\varepsilon}_{n,\mu}) = \int_{E^{\varepsilon}_{n,\mu}} \prod_{k=0}^{n-1} \frac{a_{12}(x_k, y_k)}{a(x_k, y_k, z_k)}\, d\mathbb{P}'_{xyz}(w)$$

Returning to (3.14),

$$Q^n_{m,y,z}(S(\mu,\varepsilon)) = \exp\{n\mathscr{Q}^n(\varphi+\psi)\} \int_M \left[\exp\left\{-\sum_{k=0}^{n-1} \psi(T^k(x))\right\} \times \int_{E^{\varepsilon}_{n,\mu}} \prod_{k=0}^{n-1} \frac{a_{12}(a_k, y_k)}{a(x_k, y_k, z_k)}\, d\mathbb{P}'_{xyz}(w)\right] d\mu^n(x)$$

Notice that

$$\exp\left\{-\sum_{k=0}^{n-1} \psi(T^k(x))\right\} = \exp\left\{-n\int \psi\, d\zeta^n_x\right\} \equiv \exp\{-n\langle \psi, \zeta^n_x\rangle\}$$

Moreover, in $E^{\varepsilon}_{n,\mu}$ the empirical means L_n satisfy $|\int_S \psi\, dL_n - \int_S \psi\, d\mu| < \varepsilon$, or equivalently, $|\int_M \psi\, d\zeta^n_x - \int_M \psi\, d\pi_1\mu| < \varepsilon$. Therefore,

$$Q^n_{m,y,z}(S(\mu,\varepsilon)) \geqslant \exp\{n\mathscr{Q}^n(\varphi+\psi)\} \exp\{-n\langle\psi, \pi_1\mu\rangle + \varepsilon\} \times \int_M \int_{E^{\varepsilon}_{n,\mu}} \prod_{k=0}^{n-1} \frac{a_{12}(x_k, y_k)}{a(x_k, y_k, z_k)}\, d\mathbb{P}'_{xyz}(x)\, d\mu^n(x) \tag{3.15}$$

Let us define

$$W(x,y,z) = \ln \frac{a(x,y,z)}{a_{12}(x,y)}$$

Then

$$W(V_0) + \cdots + W(V_{n-1}) = \ln \prod_{k=0}^{n-1} \frac{a(X_k, \xi_k, \xi_{k+1})}{a_{12}(X_k, \xi_k)}$$

Also,

$$I(v) = \int_S W(x,y,z)\, dv(x,y,z)$$

For each $\varepsilon' > 0$, let us define the set

$$F^{\varepsilon'}_{n,\mu} = \left[\left| \frac{\sum_{k=0}^{n-1} W(V_k)}{n} - \int_S W\, d\mu \right| < \varepsilon' \right] \tag{3.16}$$

Then, in $F^{\varepsilon'}_{n,\mu}$, we have

$$\prod_{k=0}^{n-1} \frac{a_{12}(X_k, \xi_k)}{a(X_k, \xi_k, \xi_{k-1})} > \exp\{ -n[I(\mu) + \varepsilon']\}$$

Therefore, for any $\varepsilon' > 0$, we conclude from (3.15) that

$$\begin{aligned} Q^n_{m,\,y,\,z}(S(\mu, \varepsilon)) \geqslant \exp\{n\mathscr{Q}^n(\varphi + \psi)\} \exp\{ -n[\langle \psi, \pi_1\mu\rangle + \varepsilon]\} \\ \times \exp\{ -n[I(\mu) + \varepsilon']\}\ \mathbb{P}'_{\mu^n,\,y,\,z}(E^{\varepsilon}_{n,\mu} \cap F^{\varepsilon'}_{n,\mu}) \end{aligned}$$

By taking into account that $\mathscr{Q}^n(\varphi + \psi) \to \mathscr{P}(\varphi + \psi)$ as $n \to +\infty$, we have

$$\begin{aligned} \varliminf_{n \to +\infty} \frac{1}{n} \ln Q^n_{m,\,y,\,z}(S(\mu, \varepsilon)) \geqslant \mathscr{P}(\varphi + \psi) - \int \psi\, d\pi_1\, \mu - I(\mu) - (\varepsilon + \varepsilon') \\ + \varliminf_{n \to +\infty} \frac{1}{n} \ln \mathbb{P}'_{\mu^n,\,y,\,z}(E^{\varepsilon}_{n,\mu} \cap F^{\varepsilon'}_{n,\mu}) \end{aligned}$$

But $\mathscr{P}(\varphi + \psi) = \mathscr{Q}_m(\psi)$,

$$I_m(\pi_1\mu) = \sup_{V \in C(M)} \left\{ \int V\, d\pi_1\, \mu - \mathscr{Q}_m(V) \right\}$$

and

$$\mathscr{Q}_m(\psi) - \int \psi\, d\pi_1\, \mu \geqslant -I_m(\pi_1\mu)$$

So, if we prove that

$$\lim_{n \to +\infty} \mathbb{P}'_{\mu^n,\,y,\,z}(E^{\varepsilon}_{n,\mu} \cap F^{\varepsilon'}_{n,\mu}) = 1 \tag{3.17}$$

then

$$\begin{aligned} \varliminf_{n \to +\infty} \frac{1}{n} \ln Q^n_{m,\,y,\,z}(s(\mu, \varepsilon)) &\geqslant -I_m(\pi_1\mu) - I(\mu) - (\varepsilon + \varepsilon') \\ &= -I^m(\mu) - (\varepsilon + \varepsilon') \end{aligned}$$

Moreover, from Proposition 3.1, $I^m(\nu) > I^m(\mu) - \varepsilon$. Then

$$\underline{\lim}_{n \to +\infty} \frac{1}{n} \ln Q^n_{m,\, y,\, z}(S(\mu, \varepsilon)) \geqslant -I^m(\nu) - \varepsilon'$$

Since $S(\mu, \varepsilon) \subset G$ for $\varepsilon > 0$ sufficiently small, we get

$$\underline{\lim}_{n \to +\infty} \frac{1}{n} \ln Q^n_{m,\, y,\, z}(G) \geqslant -I^m(\nu) - \varepsilon'$$

and the above inequality holds for all $\nu \in G$ and $\varepsilon' > 0$. Since ε' is arbitrary, we get

$$\underline{\lim}_{n \to +\infty} \frac{1}{n} \ln Q^n_{m,\, y,\, z}(G) \geqslant -\inf_{\nu \in G} I^m(\nu) \quad \blacksquare$$

Now it remains to prove the limit in (3.17). By taking into account the definition of μ^n and the limit in (1.13) we have for any $V \in C(M)$

$$\begin{aligned} \lim_{n \to +\infty} \frac{1}{n} \ln \int_M \exp\left\{ n \int_M V(z)\, d\zeta^n_x(z) \right\} d\mu^n(x) \\ &= \mathscr{P}(\varphi + \psi + V) - \mathscr{P}(\varphi + \psi) \\ &= \mathscr{Q}_{\mu_n}(\psi + V) - \mathscr{Q}_{\mu_n}(\psi) \equiv \tilde{\mathscr{Q}}^{\psi}(V) \end{aligned}$$

Let $\tilde{I}^{\psi}(\cdot)$ be the Legendre transform of $\tilde{\mathscr{Q}}^{\psi}(V)$. Then

$$\tilde{\mathscr{Q}}^{\psi}(V) = \sup_{\nu \in \mathscr{M}_1(M)} \left(\int V\, d\nu - \tilde{I}^{\psi}(\nu) \right)$$

so that

$$\tilde{I}^{\psi}(\nu) = \sup_{V \in C(M)} \left(\int V\, d\nu - \mathscr{P}(\varphi + \psi + V) \right) + \mathscr{P}(\varphi + \psi)$$

Since $-\int (\varphi = \psi)\, d\nu - h_\nu(T)$ is the Legendre transform of $\mathscr{P}(\varphi + \psi + V)$ in V we get

$$\tilde{I}^{\psi}(\nu) = \begin{cases} \int (\varphi + \psi)\, d\nu - h_\nu(T) + \mathscr{P}(\varphi + \psi), & \nu \in \mathscr{M}^T(M) \\ +\infty, & \text{otherwise} \end{cases}$$

However, according to the choice of μ,

$$\mathscr{P}(\varphi+\psi)=\int(\varphi+\psi)\,d\pi_1\mu+h_{\pi_1\mu}(T)$$

and $\pi_1\mu$ is the unique such measure. Hence, $\tilde{I}^{\psi}(\nu)=0$ if and only if $\nu=\pi_1\mu$. Moreover, $\tilde{I}^{\psi}(\cdot)$ is convex and lower-semicontinuous in the weak topology.

Let us define

$$I'_\mu(\nu)=\tilde{I}^{\psi}(\pi_1\nu)+I'(\nu), \qquad \nu\in\mathscr{M}_1(S)$$

where $I'(\cdot)$ is the functional in (3.9).

Lemma 3.6. $I'_\mu(\nu)=0$ if and only if $\nu=\mu$.

Proof. We know that $\tilde{I}^{\psi}(\tilde{\mu})=0$ if and only if $\tilde{\mu}=\pi_1\mu$. As in Lemma 2.5 in Donsker and Varadhan (1975), one can prove that $I'(\nu)=0$ if and only if ν is invariant for Π'. The result follows from Lemma 3.5. ∎

Let $Q'_{\mu^n,\,y,\,z}(\cdot)$ be defined by

$$Q'_{\mu^n,\,y,\,z}(A)=\mathbb{P}'_{\mu^n,\,y,\,z}[L_n\in A], \qquad A\in\mathscr{B}(\mathscr{M}_1(S))$$

Similarly to Section 2, one can prove that

$$\overline{\lim_{n\to+\infty}}\frac{1}{n}\ln Q'_{\mu^n,\,y,\,z}(F)\leqslant-\inf_{\nu\in F}I'_\mu(\nu) \tag{3.18}$$

for any closed $F\subset\mathscr{M}_1(S)$.

Proposition 3.3. $\forall\varepsilon>0$, $\forall\varepsilon'>0$,

$$\lim_{n\to+\infty}\mathbb{P}'_{\mu^n,\,y,\,z}(E^{\varepsilon}_{n,\,\mu}\cap F^{\varepsilon'}_{n,\,\mu})=1$$

where $E^{\varepsilon}_{n,\,\mu}$ and $F^{\varepsilon'}_{n,\,\mu}$ are the sets in (3.13) and (3.16) respectively.

Proof. The set $F\equiv[S(\mu,\varepsilon)]^c$ is closed in $\mathscr{M}_1(S)$. The unique minimum point of $I'_\mu(\cdot)$ is μ (see Lemma 3.6) and $\mu\notin[S(\mu,\varepsilon)]^c$. Moreover, $I'_\mu(\cdot)$ is weakly lower-semicontinuous in $\mathscr{M}_1(S)$, has compact level sets, and the upper bound in (3.18) holds. Hence, Theorem II.3.3 in ref. 4 allows one to say that there exists $N\equiv N(F)>0$ such that

$$Q'_{\mu^n,\,y,\,z}(F)\leqslant e^{-nN}$$

for all sufficiently large n. Therefore,

$$\lim_{n\to+\infty} Q'_{\mu^n,\, y,\, z}(S(\mu, \varepsilon)) = \lim_{n\to+\infty} \mathbb{P}'_{\mu^n,\, y,\, z}(E^{\varepsilon}_{n,\,\mu}) = 1$$

Now, let

$$A^{\varepsilon'} = \left\{ v \in \mathscr{M}_1(S) : \left| \int_S W\, d\mu - \int_S Q\, d\mu \right| < \varepsilon' \right\}$$

which is open in $\mathscr{M}_1(S)$. Moreover $\mu \in A^{\varepsilon'}$. Again, Theorem II.3.3 in ref. 4 implies that

$$\lim_{n\to+\infty} Q'_{\mu^n,\, t,\, z}(A^{\varepsilon'}) = \lim_{n\to+\infty} \mathbb{P}'_{\mu^n,\, y,\, z}(F^{\varepsilon'}_{n,\,\mu}) = 1 \quad \blacksquare$$

Remark 3.1. Up to now we have derived level 2 large deviations estimates for $(V_n)_{n\geqslant 0}$ in the space $(S^{\mathbb{N}}, \sigma(\mathscr{C}), \mathscr{P}_{m,\, y,\, z})$. The level-1 rate functional for the autocorrelations M_n is obtained from the Contraction Principle as one can see in ref 1.

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